

# Smooth sandwich gravitational waves

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## Abstract

Gravitational waves which are smooth and contain two asymptotically flat regions are constructed from the homogeneous  $pp$ -waves vacuum solution. Motion of free test particles is calculated explicitly and the limit to an impulsive wave is also considered.

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The widely known class of  $pp$ -waves [1] is characterized by the existence of a quadruple Debever-Penrose null vector field which is covariantly constant. In vacuum the metric can be written as

$$ds^2 = 2 d\zeta d\bar{\zeta} - 2 du dv - (f + \bar{f}) du^2 , \quad (1)$$

where  $f(u, \zeta)$  is an arbitrary function of  $u$ , analytic in  $\zeta$ . The simplest case for which the metric describes gravitational waves arises when  $f$  is of the form

$$f(u, \zeta) = d(u) \zeta^2 , \quad (2)$$

where  $d(u)$  is an *arbitrary* function of  $u$ ; such solutions are called homogeneous  $pp$  waves (or “plane” gravitational waves). Performing the transformation [2]

$$\zeta = \frac{1}{\sqrt{2}} (Px + iQy) , \quad v = \frac{1}{2} (t + z + PP'x^2 + QQ'y^2) , \quad u = t - z , \quad (3)$$

where real functions  $P(u) \equiv P(t - z)$ ,  $Q(u) \equiv Q(t - z)$  are solutions of differential equations

$$P'' + d(u) P = 0 , \quad Q'' - d(u) Q = 0 , \quad (4)$$

(here prime denotes the derivative with respect to  $u$ ) the metric can be written in the form

$$ds^2 = -dt^2 + P^2 dx^2 + Q^2 dy^2 + dz^2 , \quad (5)$$

which is more suitable for physical interpretation. Considering free test particles standing at fixed  $x$ ,  $y$  and  $z$ , their *relative* motion in the  $x$ -direction is given by the function  $P(u)$  while it is given by  $Q(u)$  in the  $y$ -direction. The motions are unaffected in the  $z$ -direction which demonstrate transversality of the waves.

Assuming “profile” functions  $d(u)$  non-vanishing on some finite interval of  $u$  only, sandwich gravitational waves were constructed in [3]-[6] and elsewhere. Here we consider the function  $d(u)$  of the form

$$d(u) = \frac{a}{\cosh^2(bu)} , \quad (6)$$

where  $a$  and  $b$  are arbitrary real positive constants. Since all derivatives of  $d(u)$  are continuous the corresponding gravitational waves are *smooth*, contrary to “standard” sandwich waves explicitly presented in literature. On the other hand, the space-times (1) given by Eqs. (2) and (6) do not contain flat regions in front of the wave and behind it. They are curved everywhere, becoming flat only *asymptotically* as  $u \rightarrow \pm\infty$ . Therefore, we should call them “smooth asymptotic sandwich waves”.

In order to find the form (5) of the metric exhibiting naturally the particle motions we solve the equations (4). Introducing a substitution

$$\xi = -\tanh(bu) , \quad (7)$$

the equations take the form

$$(1 - \xi^2) \frac{d^2 R}{d\xi^2} - 2\xi \frac{dR}{d\xi} \pm \frac{a}{b^2} R = 0 , \quad (8)$$

where the upper sign is applied for  $R = P$  and the lower sign for  $R = Q$ . Clearly, a general solution can be written as a linear combination of the Legendre functions of the first kind  $P_\alpha(\xi)$  and the second kind  $Q_\alpha(\xi)$  where  $\alpha = (\sqrt{1 \pm 4a/b^2} - 1)/2$ . It is natural to impose the conditions  $R(u \rightarrow -\infty) \rightarrow 1$  and  $R'(u \rightarrow -\infty) \rightarrow 0$  so that the metric (5) is written in explicit Minkowski form in the asymptotic region where  $u \rightarrow -\infty$ . These conditions are satisfied by particular solutions of the form

$$P(u) = P_\mu(\xi(u)) , \quad Q(u) = P_\nu(\xi(u)) , \quad (9)$$

where

$$\mu = (\sqrt{1 + 4a/b^2} - 1)/2 , \quad \nu = (\sqrt{1 - 4a/b^2} - 1)/2 . \quad (10)$$

Typical behavior of the functions  $P, Q$  is shown in Fig. 1. It can be observed that in both asymptotic regions  $u \rightarrow \pm\infty$  the particles move uniformly. This can be shown analytically using the definition  $P_\alpha(\xi) \equiv F(-\alpha, \alpha+1; 1; (1-\xi)/2)$  where  $F$  is the hypergeometric function. Clearly,  $P \rightarrow 1, Q \rightarrow 1$  as  $u \rightarrow -\infty$ . For  $u \rightarrow +\infty$  (corresponding to  $\xi \rightarrow -1$ ) we use the identity 15.3.10. in [7] according to which

$$F\left(-\alpha, \alpha+1; 1; \frac{1-\xi}{2}\right) = -\frac{\sin \pi \alpha}{\pi} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \times \left[2\psi(n+1) - \psi(n-\alpha) - \psi(\alpha+n+1) - \ln\left(\frac{1+\xi}{2}\right)\right] \left(\frac{1+\xi}{2}\right)^n, \quad (11)$$

where  $\psi$  is the digamma function. Using the identity  $\psi(-\alpha) = \psi(1-\alpha) + 1/\alpha$  and the limit  $\ln[(1+\xi)/2] \rightarrow -2bu$  for  $u \rightarrow +\infty$  we conclude that

$$P(u \rightarrow +\infty) = C_1 + C_2 u, \quad Q(u \rightarrow +\infty) = D_1 + D_2 u, \quad (12)$$

where

$$C_1 = \frac{\sin \pi \mu}{\pi} \left[ \frac{1}{\mu} + \psi(1+\mu) + \psi(1-\mu) - 2\psi(1) \right], \quad C_2 = -\frac{2b}{\pi} \sin \pi \mu, \\ D_1 = \frac{\sin \pi \nu}{\pi} \left[ \frac{1}{\nu} + \psi(1+\nu) + \psi(1-\nu) - 2\psi(1) \right], \quad D_2 = -\frac{2b}{\pi} \sin \pi \nu. \quad (13)$$

For particular values of  $\mu$  corresponding to  $\mu = m = 0, 1, 2, \dots$ , the constant  $C_2$  vanishes so that  $P(u \rightarrow +\infty) = C_1$ . These cases coincide with solutions  $P(u)$  given by Legendre *polynomials*  $P_m(\xi)$  and we easily get  $C_1 = P_m(\xi = -1) = (-1)^m$ . Explicit particular solutions of this type arise when  $a/b^2 = m(m+1)$  so that  $P(u) = -\tanh(\sqrt{a/2}u)$  for  $m = 1$ ,  $P(u) = [3 \tanh^2(\sqrt{a/6}u) - 1]/2$  for  $m = 2$ , etc. Note also that there are no analogous solutions for  $Q(u)$  since (10) admits  $\nu \in \langle -1, 0 \rangle$  only. Therefore, the only non-negative integer (for which the constant  $D_2$  vanishes) is  $\nu = 0$  representing a trivial case  $d(u) = 0$ . For all other values of  $\nu$  the constant  $D_2$  is positive so that  $Q(u) \rightarrow +\infty$  as  $u \rightarrow +\infty$ .

We can also use the above results for construction of impulsive gravitational waves. The sequence of smooth functions (6) approach the  $\delta$  function (in a distributional sense) as  $a \rightarrow 0$  if the second parameter is  $b = 2a$  (so that the normalization condition  $\int_{-\infty}^{+\infty} d(u) du = 1$  holds for arbitrary  $a$ ). Considering this limit, indicated also in Fig. 1,  $\alpha = (\sqrt{1 \pm 1/a} - 1)/2 \rightarrow 0$  and using Eqs. (13) we get

$$P(u) = 1 - u \Theta(u), \quad Q(u) = 1 + u \Theta(u), \quad (14)$$

where  $\Theta$  is the Heaviside step function ( $\Theta = 0$  for  $u < 0$ ,  $\Theta = 1$  for  $u > 0$ ). Therefore, particle motion in the impulsive gravitational wave is the same as in the limit of “standard” sandwich waves [2].

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## Figure Caption

Fig. 1. Typical behavior of the functions  $P(u)$  and  $Q(u)$  determining relative motion of free test particles (initially at rest) in  $x$  and  $y$ -directions, respectively, caused by smooth sandwich gravitational waves for different values of  $a = \frac{1}{48}, \frac{2}{48}, \dots, \frac{12}{48}$ , with  $b = 2a$ . The values  $a = \frac{1}{8}, \frac{1}{24}, \frac{1}{48}$  correspond to  $\mu = m = 1, 2, 3$ , respectively, for which  $P \rightarrow (-1)^m$  as  $u \rightarrow +\infty$ .

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